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On the Non-Euclidean Trigonometry.

BY WILLIAM E. STORY.

In the "Sixth memoir upon quantics"* Professor Cayley has given a projective definition of geometrical quantity of one dimension (distance between two points and angle between two lines), which has been generalized by Dr. Klein in a paper "Ueber die sogenannte Nicht-Euklidische Geometrie."† Professor Cayley afterwards gave an example of the non-Euclidean trigonometry in the plane, obtaining the formulae for the special case in which the fundamental conic or "absolute" is a circle, and the constants of measurement have particular values.‡ I propose here to deduce the formulae for the general case of projective measurement in the plane with an arbitrary absolute and arbitrary constants. The application of the results to a space of two dimensions with constant curvature in a third dimension is evident. It is also evident from the nature of the projective measurement that the results are independent of the particular choice of coördinates, and therefore are not affected by the circumstance that a real point may have imaginary coördinates.

Following Professor Cayley's notation, I put a dash over any quantity measured projectively. The projective measure $\bar{a}$ of the distance between two points B and C , and the projective measure $\bar{A}$ of the angle between two lines b and c , are defined thus:

$$\bar{a} = k \ln \alpha, \quad \bar{A} = k' \ln \alpha',$$

where α is the anharmonic ratio of B and C with respect to the intersections of the line BC with the absolute, and α' is the anharmonic ratio of b and c with

* Philosophical Transactions, Vol. 149, 1859. † Mathematische Annalen, Vol. IV, pp. 573-625.

‡ Mathematische Annalen, Vol. V, pp. 630-634.

respect to the tangents from the point bc to the absolute, and k and k' are two arbitrary constants corresponding to the arbitrary units of ordinary linear and angular measurement.

Let the equations of the absolute in point-coördinates x, y, z , and in tangential coördinates u, v, w , be

$$\Omega \equiv f(x, y, z) = 0 \quad \text{and} \quad \mathfrak{U} \equiv F(u, v, w) = 0,$$

respectively; and let

$$\begin{aligned} \Omega_{11} &\equiv f(x_1, y_1, z_1), & \Omega_{12} &\equiv \frac{1}{2} \left[x_1 \frac{\partial \Omega_{23}}{\partial x_2} + y_1 \frac{\partial \Omega_{23}}{\partial y_2} + z_1 \frac{\partial \Omega_{23}}{\partial z_2} \right], \\ \mathfrak{U}_{11} &\equiv F(u_1, v_1, w_1), & \mathfrak{U}_{12} &\equiv \frac{1}{2} \left[u_1 \frac{\partial \mathfrak{U}_{23}}{\partial u_2} + v_1 \frac{\partial \mathfrak{U}_{23}}{\partial v_2} + w_1 \frac{\partial \mathfrak{U}_{23}}{\partial w_2} \right], \end{aligned}$$

and similarly for other combinations of the suffices 1, 2, 3.

Let A, B, C be the vertices of a triangle, a, b, c their sides, $\overline{A}, \overline{B}, \overline{C}$, $\overline{a}, \overline{b}, \overline{c}$ the projective measures of the angles and opposite sides, respectively; and let the coördinates of A, B, C, a, b, c , be $x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; u_1, v_1, w_1; u_2, v_2, w_2; u_3, v_3, w_3$, respectively. Then, by the above definitions,

$$\overline{a} = k \ln \frac{\Omega_{23} + \sqrt{\Omega_{23}^2 - \Omega_{22}\Omega_{33}}}{\Omega_{23} - \sqrt{\Omega_{23}^2 - \Omega_{22}\Omega_{33}}},$$

$$\overline{A} = k' \ln \frac{\mathfrak{U}_{23} + \sqrt{\mathfrak{U}_{23}^2 - \mathfrak{U}_{22}\mathfrak{U}_{33}}}{\mathfrak{U}_{23} - \sqrt{\mathfrak{U}_{23}^2 - \mathfrak{U}_{22}\mathfrak{U}_{33}}},$$

which can be put into the forms

$$\overline{a} = 2ik \cos^{[-1]} \frac{\Omega_{23}}{\sqrt{\Omega_{22}\Omega_{33}}}, \quad \overline{A} = 2ik' \cos^{[-1]} \frac{\mathfrak{U}_{23}}{\sqrt{\mathfrak{U}_{22}\mathfrak{U}_{33}}},$$

$$i. e. \quad \cos \left(\frac{\overline{a}}{2ik} \right) = \frac{\Omega_{23}}{\sqrt{\Omega_{22}\Omega_{33}}}, \quad \cos \left(\frac{\overline{A}}{2ik'} \right) = \frac{\mathfrak{U}_{23}}{\sqrt{\mathfrak{U}_{22}\mathfrak{U}_{33}}}.$$

For convenience I take for the lines $x = 0, y = 0, z = 0$ respectively two conjugate polars through the point A and the polar of A with respect to Ω , so that

$$\Omega \equiv x^2 + y^2 + z^2, \quad \text{and hence} \quad \mathfrak{U} \equiv u^2 + v^2 + w^2,$$

$$\Omega_{23} \equiv x_2x_3 + y_2y_3 + z_2z_3, \quad \mathfrak{U}_{23} \equiv u_2u_3 + v_2v_3 + w_2w_3,$$

$$x_1 : y_1 : z_1 = 0 : 0 : 1,$$

$$u_3 : v_3 : w_3 = y_2 : -x_2 : 0, \quad u_2 : v_2 : w_2 = y_3 : -x_3 : 0;$$

and, putting $x_2 = z_2 r_2 \cos \phi_2$, $y_2 = z_2 r_2 \sin \phi_2$, $x_3 = z_3 r_3 \cos \phi_3$, $y_3 = z_3 r_3 \sin \phi_3$, I find

$$\begin{aligned}\Omega_{11} &= 1, \quad \Omega_{22} = z_2^2 (r_2^2 + 1), \quad \Omega_{33} = z_3^2 (r_3^2 + 1), \\ \Omega_{12} &= z_2, \quad \Omega_{13} = z_3, \quad \Omega_{23} = z_2 z_3 [r_2 r_3 \cos (\phi_3 - \phi_2) + 1], \\ \mathcal{U}_{22} &= z_2^2 r_2^2, \quad \mathcal{U}_{33} = z_3^2 r_3^2, \quad \mathcal{U}_{23} = z_2 z_3 r_2 r_3 \cos (\phi_3 - \phi_2), \\ \cos \left(\frac{\bar{a}}{2ik} \right) &= \pm \frac{1 + r_2 r_3 \cos (\phi_3 - \phi_2)}{\sqrt{(r_2^2 + 1)(r_3^2 + 1)}}, \quad \cos \left(\frac{\bar{A}}{2ik'} \right) = \pm \cos (\phi_3 - \phi_2), \\ \cos \left(\frac{\bar{b}}{2ik} \right) &= \frac{\pm 1}{\sqrt{r_2^2 + 1}}, \quad \sin \left(\frac{\bar{b}}{2ik} \right) = \frac{\pm r_2}{\sqrt{r_2^2 + 1}}, \\ \cos \left(\frac{\bar{c}}{2ik} \right) &= \frac{\pm 1}{\sqrt{r_3^2 + 1}}, \quad \sin \left(\frac{\bar{c}}{2ik} \right) = \frac{\pm r_3}{\sqrt{r_3^2 + 1}}, \\ \cos \left(\frac{\bar{a}}{2ik} \right) &= \pm \cos \left(\frac{\bar{b}}{2ik} \right) \cos \left(\frac{\bar{c}}{2ik} \right) \pm \sin \left(\frac{\bar{b}}{2ik} \right) \sin \left(\frac{\bar{c}}{2ik} \right) \cos \left(\frac{\bar{A}}{2ik'} \right),\end{aligned}$$

where each sign $\pm$ has to be so determined that the distance between two coincident points and the angle between two coincident lines shall be 0, or some multiple of the whole length of a straight line and whole angle about a point respectively, while it remains arbitrary in which of the two possible directions the measure is positive. If the points B and C coincide, $\bar{c} = \bar{b}$, $\bar{a} = 0$, $\bar{A} = 0$, hence, by the last formula,

$$1 = \pm \cos^2 \left(\frac{\bar{b}}{2ik} \right) \pm \sin^2 \left(\frac{\bar{b}}{2ik} \right),$$

therefore the upper sign is to be given to both terms, *i. e.*

$$\cos \left(\frac{\bar{a}}{2ik} \right) = \cos \left(\frac{\bar{b}}{2ik} \right) \cos \left(\frac{\bar{c}}{2ik} \right) + \sin \left(\frac{\bar{b}}{2ik} \right) \sin \left(\frac{\bar{c}}{2ik} \right) \cos \left(\frac{\bar{A}}{2ik'} \right),$$

corresponding to the formula of spherical trigonometry

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

from which, as is well known, all the other formulae of spherical trigonometry can be obtained, without the use of any other relations than those implied in the definitions of the trigonometric functions. Hence, in a uniformly curved space of two dimensions, with a projective measurement based upon any conic, whose constants of linear and angular measurement are k and k' respectively, *the trigonometrical formulae will be obtained from those of spherical trigonometry by replacing each side (or arc) by the corresponding side (or straight line) divided by*

$2ik$, and each angle by the corresponding angle divided by $2ik'$. Of course, what we call a straight line in such a curved space is a geodesic line, *i. e.* the shortest line between any two of its points measured as above.

If $k = \frac{1}{2}$, $k' = \frac{1}{2i}$,

then $\frac{\bar{a}}{2ik} = -i\bar{a}$, $\frac{\bar{A}}{2ik'} = \bar{A}$,

$$\sin\left(\frac{\bar{a}}{2ik}\right) = -i \sinh \bar{a}, \quad \cos\left(\frac{\bar{a}}{2ik}\right) = \cosh \bar{a},$$

whence

$$\cos \bar{A} = - \frac{\cosh \bar{a} - \cosh \bar{b} \cosh \bar{c}}{\sinh \bar{b} \sinh \bar{c}},$$

the formula given by Professor Cayley, which, however, holds for any other fundamental conic, as well as for a circle.

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